THICK ORTHOTROPIC RECTANGULAR PLATES ON ELASTIC FOUNDATIONS

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ABSTRACT

In this research, Mindlin's thick plate theory is extended to include orthotropic plates under the effects of externally distributed moments and shearing forces at top and bottom faces of the plate. These shearing forces produce in-plane forces in plates and the extensional effects of these in-plane forces are considered. The transverse sections of the plates have five degrees of freedom. These are the transverse deflection, the two independent rotations of the normal to the middle plane and the two mutually perpendicular membrane displacements. Thus, five expressions of the governing equations for thick orthotropic plates are obtained with the inclusion of the effects of externally distributed moments and applied shearing forces.

As an application to the generation of distributed moments and shearing forces, the problems of thick orthotropic plates resting on elastic foundations with both compressional and frictional restraints are investigated.

The finite-difference method was used to solve the governing equations. Besides, finite elements are formulated and used. Good agreements are found in the results from both methods of solution.

KEYWORDS

Elastic foundations, Finite differences, Finite elements, Orthotropic plate, Thick plates.

NOTATIONS

Symbol	Descriptions
c^2	Correction factor for transverse shear
D_x , D_y	Flexural rigidities of orthotropic plates in x and y directions
E_x, E_y, E_z	Moduli of elasticity of orthotropic plates in x, y, and z-direction
F_x, F_y	Horizontal friction forces in x and y-directions
G_{xy}, G_{xz}, G_{yz}	Shearing moduli for xy, xz, and yz- plane respectively
h	Plate thickness
K_x, K_y, K_z	Moduli of subgrade reactions in x, y, and z-directions
N_x, N_y, N_{xy}	Membrane forces
P(x,y)	Soil reactions
Q_x, Q_y	Transverse shearing force per unit width
q(x,y)	Transverse load per unit area
u, v	Displacement in x and y-directions
u_o, v_o	Displacement of the middle plane of the plate in *x* and *y*-directions.
w	Displacement in z –direction
x, y, z	Original axes
δ_x, δ_y	Angles of friction of soil in x and y-direction
ε_x, ε_y, ε_z	Normal strain in x, y, and z-direction
$\varepsilon_{x^\circ y^\circ}$	Shearing deformation due to membrane forces

Symbol	Descriptions
$\nu_{ij} = -\varepsilon_j/\varepsilon_i$	Poisson's ratio of compressive strength in j-direction to the tensile strain in i-direction when only tensile stress σ_i is acting along the i-axis (for orthotropic materials)
Ψ_x, Ψ_y	Rotation of the transverse section in xz and yz directions
γ_{xy}, γ_{yz}, γ_{xz}	Engineering shearing strains in xy, yz, and xz-planes
τ_{xy}, τ_{yz}, τ_{xz}	Shearing stresses in xy, yz, and xz- planes
σ_x, σ_y, σ_z	Normal stresses in x, y, and z-directions

INTRODUCTION

An exact theory for analysis of plates should be derived from the three-dimensional elasticity. Due to the complexity of the problem, the following simplifying assumptions are made in the classical theories of thin plates[1]

1. Plane transverse sections before bending will remain plane after bending (linear strain distribution in a cross section).
2. Normal lines to the middle plane will remain straight and normal to the deflected middle plane (no transverse shearing deformations).
3. Normal strains in the normal lines to the middle plane are neglected (no change in thickness)
4. The deformations are small (linear theory of small deformations).
5. Linear stress - strain relationship is assumed for the material of the plate (Hooke's laws)

According to these assumptions, the behavior of the plate under transverse loads is characterized by one deformation function which is the deflection of the middle plane (the transverse section has one degree of freedom which is the deflection w=w(x,y))

To develop better formulations, restraints from one or more assumptions in the classical theory must be removed.

Reissner [2] and Mindlin [3] derived the governing equations for bending of thick plates by allowing the line normal to the middle plane to rotate independent of the slopes of the middle plane. The transverse shearing deformations are thus considered. The behavior of the plate under transverse loads will be characterized by three independent functions which are the transverse deflection of the middle plane w(x,y) and the two rotations ($\Psi_x(x,y)$ and $\Psi_y(x,y)$) of the normal to the middle plane in the planes of xz and yz respectively.

In Reissner-Mindlin theory, the cross section is assumed to remain plane after bending (no warping). To account for this incompatible deformation, a shear correction factor (c^2) is introduced in the main governing equations which is a numerical factor representing the restraint of cross section against warping, commonly assumed to be 5/6 for rectangular sections.

Schmidt [4] and Levinson [5] removed the restrictions from the second assumption in the classical theory by allowing the cross sections to rotate and warp in such a fashion that they remain normal to the shear-free top and bottom surfaces. The theory is extended further more to include the capability of the plate to take external shearing forces and moments [6].

In this paper, the original Mindlin's thick plate theory is extended to include thick orthotropic plates. The effects of applied shearing forces at top and bottom faces of the plate are included in the plate. Thus, a cross section will have five degrees of freedom which are the lateral deflection w = w(x,y), two rotations of the normal to the middle plane ($\Psi_x = \Psi_x(x,y)$ and $\Psi_y = \Psi_y(x,y)$)and the membrane displacements in the two perpendicular directions in the middle plane of the plate ($u_o = u_o(x,y)$ and $v_o = v_o(x,y)$).

ORTHOTROPIC MATERIALS

Elastic materials under stresses are divided according to the types of induced deformations [5-6]:

1. Anisotropic Materials: They have different elastic properties in different directions. There are (21) elastic constants to describe the linear stress-strain relations (Hooke's law). The application of one type of stress (either normal or shear stress) leads to two types of deformations (axial and shear deformations) in the same time.

2. Isotropic Materials: They have the same elastic properties in all directions and deform in one type of deformations (axial deformations with axial stresses and shear deformations with shear stresses). Only two independent elastic constants describe the linear relations between stresses and strains.

3. Orthotropic Materials: They have three planes of symmetry which are mutually perpendicular. Thus, they have different elastic properties in orthogonal directions (or principal directions). The orthotropic materials behave like isotropic materials if the loads are applied in principal directions and as anisotropic materials otherwise. In principal directions, normal stresses produce only normal strains and shearing stresses only shearing strains. Nine independent elastic constants describe the stress-strain relations.

ORTHOTROPIC PLATES

A plate may be considered orthotropic if it has different elastic properties or different moments of inertia in orthogonal directions.

There are various types of orthopic plate

1. Plates made from naturally orthotropic materials.
2. Plates made from different materials such as concrete slabs reinforced by different amounts of reinforcement in different directions.
3. Stiffened plates which can be transformed to equivalent orthotropic plates, such as ribbed slabs.

FORMULATION

In the following analysis, a thick orthotropic plate of uniform thickness is considered.

The coordinate plane xy coincides with the middle plane of the plate and the z-axis is the upward normal to the middle plane. Thus, the upward deflections are considered positive. A rectangular plate element of sides (dx.dy) and thickness (h) is under transverse distributed loads q=q(x,y) and shearing stresses on the top and bottom faces $\tau_{ZX(\pm h/2)}$ and $\tau_{zy\ (\pm h\ /2)}$. Besides, distributed moments $\mu_x = \mu_x(x,y)$ and $\mu_y = \mu_y(x,y)$ (per unit area) may be acting on the plate, Fig.(1).

The behavior of the plate under the applied loads is formulated according to the following assumptions:

1. Plane cross sections will remain plane after bending (no warping).
2. The cross sections will have additional rotations due to the transverse shearing forces. Warping of cross sections by these forces is considered through a correction factor.

3. The normal line to the middle plane has constant length (ε_z=0).

KINEMATICS CONSIDERATIONS

A cross section in xz-plane before and after deformation is shown in Fig. (2).

The normal line to the middle plane has five degrees of freedom (deflection w, two rotations (Ψ_x and Ψ_y) and two in-plane displacements (u_o and v_o)).

The displacement in x-direction (u) at a point at distance z above the middle plane will be:

$$u = u_o + z\psi_x \quad \text{...(1)}$$

where

$u_o = u_o(x,y)$ is the displacement at the middle plane

$\Psi_x = \Psi\ (x,y)$ is the rotation of the normal line in clockwise direction). Also, the transverse deflection w is:

$$w = w\,(x,y) \quad \text{(independent of z)} \quad \text{...(2)}$$

Similarly, for yz-plane:

$$v = v_o + z\Psi_x \quad \text{...... .... (3)}$$

The mathematical expressions for strains are:

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\partial u_o}{\partial x} + z\frac{\partial \psi_x}{\partial x} = \varepsilon_{xo} + z\frac{\partial \psi_x}{\partial x} \quad \text{...................................... (4)}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = \frac{\partial v_o}{\partial y} + z\frac{\partial \Psi_y}{\partial y} = \varepsilon_{yo} + z\frac{\partial \Psi_y}{\partial y} \quad \text{.......................................(5)}$$

From Eqs. (4) and (5), it is noticed that ε_x and ε_y are linear in z (plane cross section assumption). Also,

$$\varepsilon_z = \frac{\partial w}{\partial z} = 0 \quad \text{............................ .... (6)}$$

The engineering shearing strains are:

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\gamma_{xy} = \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} + (\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x})z$$

$$\gamma_{xy} = \gamma_{x_o y_o} + (\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x})z \quad \ldots\ldots (7)$$

$$\gamma_{zx} = \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \psi_x + \frac{\partial w}{\partial x} \quad \ldots\ldots (8)$$

$$\gamma_{yz} = \gamma_{zy} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \psi_y + \frac{\partial w}{\partial y} \quad \ldots\ldots (9)$$

Using the stress-strain relations for the orthotropic materials [7] and substituting the above expressions of strains in the stress-strain relations, then:

$$\sigma_x = \frac{E_x}{1-\nu_{xy}\nu_{yx}}\left[\frac{\partial u_o}{\partial x} + \nu_{yx}\frac{\partial v_o}{\partial y}\right] + \frac{E_x}{1-\nu_{xy}\nu_{yx}}\left[\frac{\partial \psi_x}{\partial x} + \nu_{yx}\frac{\partial \psi_y}{\partial y}\right]z \quad \ldots\ldots (10)$$

$$\sigma_y = \frac{E_y}{1-\nu_{xy}\nu_{yx}}\left[\varepsilon_y + \nu_{xy}\varepsilon_x\right] \quad \ldots\ldots (11)$$

$$\sigma_y = \frac{E_y}{1-\nu_{xy}\nu_{yx}}\left[\frac{\partial v_o}{\partial y} + \nu_{xy}\frac{\partial u_o}{\partial x}\right] + \frac{E_y}{1-\nu_{xy}\nu_{yx}}\left[\frac{\partial \psi_y}{\partial y} + \nu_{xy}\frac{\partial \psi_x}{\partial x}\right]z$$

where E_x and E_y are the elastic moduli in x and y-directions and v_{yx} is Poisson's ratio of compressive strain in x-direction to the tensile strain in y-direction when only a tensile stress σ_y is acting along the y axis (for orthotropic materials). Similarly v_{xy} is defined.

Here, the normal stress σ_z in the z-direction is disregarded. Also, the shearing stresses are:

$$\tau_{xy} = \tau_{yx} = G_{xy}\gamma_{xy}\,, \tau_{xy} = G_{xy}\gamma_{x_o y_o} + G_{xy}\left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x}\right)z \qquad \text{……………} \quad (12)$$

$$\tau_{zx} = \tau_{xz} = G_{xz}\gamma_{xz}$$

$$\tau_{xz} = G_{xz}\left(\psi_x + \frac{\partial w}{\partial x}\right) \qquad \text{……………………….}(13)$$

$$\tau_{zy} = \tau_{yz} = G_{yz}\gamma_{yz}$$

$$\tau_{yz} = G_{yz}\left(\psi_y + \frac{\partial w}{\partial y}\right) \qquad \text{………………………} \quad (14)$$

The stress resultants are the two bending moments M_x and M_y, the twisting moments $M_{xy}=M_{yx}$ transverse shearing forces Q_x and Q_y (in yz and zx-planes), and the in-plane forces N_x, N_y and N_{xy} (all per unit width). From Fig. (3), the stress resultants are calculated as follows:

$$M_x = \int_{-h/2}^{h/2} z\sigma_x (1.dz) \qquad \text{…………………......} \quad (15)$$

By substituting equation (10) in the above equation and integrating, the obtained expression for the bending moment is given as

$$M_x = D_x\left(\frac{\partial \psi_x}{\partial x} + \nu_{yx}\frac{\partial \psi_y}{\partial y}\right) \qquad \text{……………………} \quad (16)$$

where

$$D_x = \frac{E_x h^3}{12(1-\nu_{xy}\nu_{yx})} \qquad \text{……………………}(17)$$

D_x is the flexural rigidity in x-direction of the yz-section. By same manner,

$$M_y = D_y\left(\frac{\partial\psi_y}{\partial y} + v_{xy}\frac{\partial\psi_x}{\partial x}\right) \quad \text{.......................(18)}$$

where

$$D_y = \frac{E_x h^3}{12(1-\nu_{xy}\nu_{yx})} \quad \text{........................(19)}$$

D_y is the flexural rigidity in y-direction of the xz-section.

The twisting moment is,

$$M_{xy} = \int_{-h/2}^{h/2} z\tau_{xy}(1.dz) \quad \text{..........................(20)}$$

By substituting equation (12) in the above equation and integrating, then

$$M_{xy} = D_{xy}\left(\frac{\partial\psi_x}{\partial y} + v_{xy}\frac{\partial\psi_y}{\partial x}\right) \quad \text{.... ....(21)}$$

where

$$D_{xy} = G_{xy}\frac{h^3}{12} \quad \text{...............................(22)}$$

D_{xy} is the torsional rigidity of the xz or yz-section.

Transverse shearing forces in transverse sections are obtained by integrating the shearing stresses over the transverse area per unit width

$$Q_x = \int_{-h/2}^{h/2} \tau_{xz}(1.dz) \quad \text{....................................(23)}$$

Substituting equation (13) in the above equation and integrating,

$$Q_x = c^2 G_{xz} h\left(\psi_x + \frac{\partial w}{\partial x}\right) \quad \text{..................................(24)}$$

where c^2 is a shear correction factor.

By the same manner, the expression for Q_y is:

$$Q_y = c^2 G_{yz} h\left(\psi_y + \frac{\partial w}{\partial y}\right) \quad \text{................................ (25)}$$

The shear correction factor c^2 is a numerical factor representing the restraint of the cross section against warping (commonly assumed to be 5/6 for rectangular sections). This correction factor considers the actually variable shearing stress (τ_{xz} or τ_{yz}) in a transverse section as uniformly distributed.

The in-plane forces per unit width are

$$N_x = \int_{-h/2}^{h/2} \sigma_x (1.dz) \tag{26}$$

By substituting Eq. (10) and after integrating,

$$N_x = \frac{E_x h}{1-\nu_{xy}\nu_{yx}} \left(\frac{\partial u_o}{\partial x} + \nu_{yx} \frac{\partial v_o}{\partial y} \right) \tag{27}$$

Also

$$N_y = \frac{E_y h}{1-\nu_{xy}\nu_{yx}} \left(\frac{\partial v_o}{\partial y} + \nu_{xy} \frac{\partial u_o}{\partial x} \right) \tag{28}$$

$$N_{xy} = N_{yx} = \int_{-h/2}^{h/2} \tau_{xy} (1.dz) = G_{xy} A \gamma_{x^o y^o}$$

$$N_{xy} = G_{xy} h \left(\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} \right) \tag{29}$$

Static Considerations

An element (dx.dy.h) is considered. Bending and twisting moments, transverse shearing and in-plane forces and the general external loads on this element are shown in Fig. (4).

By equilibrium of forces in z-direction,

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial x} + q = 0 \tag{30}$$

By equilibrium of moments in xz-plane and yz-plane,

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial x} - Q_x + \mu_x = 0 \tag{31}$$

$$\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y + \mu_y = 0 \tag{32}$$

Equilibrium of forces in x and y-directions give,

$$\frac{\partial N_x}{\partial x}+\frac{\partial N_{yx}}{\partial y}+\tau_{zx(h/2)}-\tau_{zx(-h/2)}=0 \qquad \ldots\ldots (33)$$

$$\frac{\partial N_y}{\partial y}+\frac{\partial N_{xy}}{\partial x}+\tau_{zy(h/2)}-\tau_{zy(-h/2)}=0 \qquad \ldots\ldots (34)$$

In the above equilibrium equations, (μ_x and μ_y) are considered to be the moments (per unit area) about the middle plane. If these moments are due to the applied shearing forces on the top and bottom faces, they can be calculated as:

$$\mu_x=\frac{h}{2}\left(\tau_{zx(h/2)}-\tau_{zx(-h/2)}\right) \qquad \ldots\ldots (35)$$

$$\mu_y=\frac{h}{2}\left(\tau_{zy(h/2)}-\tau_{zy(-h/2)}\right) \qquad \ldots\ldots (36)$$

The above **five** equilibrium equations (Eqs. (30) to (34)) contain **eight** unknowns (M_x , M_y, M_{xy} = M_{yx}, Q_x, Q_y , N_x , N_y and N_{xy} = N_{yx}). Thus, the problem is statically indeterminate. Additional equations are needed from compatibility of deformations (or stress resultant equations in terms of displacements, Eqs. (16), (18), (21), (24), (25), (27), (28) and (29)).

Governing Equations

The governing equations can be obtained by substituting the expressions of the stress resultants in terms of the displacements (*w* , Ψ_x, Ψ_y, u_o and v_o) in the equilibrium equations. Substitution of equations (24) and (25) into Eq. (30) gives the *first* governing equation,

$$c^2G_{xz}h\left(\frac{\partial \psi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+c^2G_{yz}h\left(\frac{\partial \psi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)+q=0 \qquad \ldots\ldots (37)$$

Also, substitution of Eqs. (16), (21) and (24) in Eq. (31) gives the *second* governing equation,

$$D_x\left(\frac{\partial^2\psi_x}{\partial x^2}+v_{yx}\frac{\partial^2\psi_y}{\partial x\partial y}\right)+D_{xy}\left(\frac{\partial^2\psi_x}{\partial y^2}+v_{yx}\frac{\partial^2\psi_y}{\partial x\partial y}\right)-c^2G_{xz}h\left(\psi_x+\frac{\partial w}{\partial x}\right)+\mu_x=0 \qquad (38)$$

Substitution of Eqs. (18), (21) and (25) in Eq. (32) gives the *third* governing equation

$$D_y\left(\frac{\partial^2\psi_y}{\partial y^2}+\nu_{xy}\frac{\partial^2\psi_x}{\partial x\partial y}\right)+D_{xy}\left(\frac{\partial^2\psi_x}{\partial y\partial x}+\frac{\partial^2\psi_y}{\partial x^2}\right)-c^2G_{yz}h\left(\psi_y+\frac{\partial w}{\partial y}\right)+\mu_y=0 \quad (39)$$

Substitution of Eqs. (27) and (29) in Eq. (33) gives *fourth* governing equation

$$\frac{E_x h}{1-\nu_{xy}\nu_{yx}}\left(\frac{\partial^2 u_o}{\partial x^2}+\nu_{yx}\frac{\partial^2 v_o}{\partial x\partial y}\right)+G_{xy}h\left(\frac{\partial^2 u_o}{\partial y^2}+\frac{\partial^2 v_o}{\partial x\partial y}\right)+\tau_{zx(h/2)}-\tau_{zx(-h/2)}=0 \quad ..(40)$$

Finally, substitution of Eqs. (28) and (29) in Eq. (34) gives the *fifth* governing equation

$$\frac{E_y h}{1-\nu_{xy}\nu_{yx}}\left(\frac{\partial^2 v_o}{\partial y^2}+\nu_{xy}\frac{\partial^2 u_o}{\partial x\partial y}\right)+G_{xy}h\left(\frac{\partial^2 u_o}{\partial x\partial y}+\frac{\partial^2 v_o}{\partial x^2}\right)+\tau_{zy(h/2)}-\tau_{zy(-h/2)}=0 \quad \ldots(41)$$

Boundary Conditions

Five natural conditions exist on a boundary edge of a plate in bending and extension. The tangent and the normal to an edge are written as (t) and (n).

i) Simply Supported Edge

(a)Roller Supported Edge

$w = 0$

$\Psi_t = 0$

$M_n = 0 \longrightarrow \frac{\partial\psi_n}{\partial n}=0$

$N_n = 0 \longrightarrow \frac{\partial u_{no}}{\partial n}-\nu_{tn}+\frac{\partial u_{to}}{\partial t}$ (42)

$u_{to} = 0$ (displacement tangent to the edge)

An alternative to $\Psi_t = 0$ is $M_{nt} = 0$. In this case

$$\frac{\partial\psi_n}{\partial t}+\frac{\partial\psi_t}{\partial n}=0$$

An alternative to $u_{to} = 0$ is $N_{nt} = 0$. In this case

$$\frac{\partial u_{no}}{\partial t} = -\frac{\partial u_{to}}{\partial n}$$

(b) Hinged Edge

$$\left.\begin{aligned} & w = 0 \\ & \Psi_t = 0 \\ & M_n = 0 \longrightarrow \frac{\partial \psi_n}{\partial n} = 0 \end{aligned}\right] \quad (43)$$

$u_{no} = 0$ (displacement normal to the edge)

$u_{to} = 0$ (displacement tangent to the edge)

An alternative to $\Psi_t = 0$ is $M_{nt} = 0$. In this case

$$\frac{\partial \psi_n}{\partial t} + \frac{\partial \psi_t}{\partial n} = 0$$

ii) Clamped Edge

$w = 0$

$$\left.\begin{aligned} & \frac{\partial w}{\partial n} = 0 \text{ (zero normal shape)} \\ & \Psi_t = 0 \\ & u_{no} = 0 \text{ (displacement normal to the edge)} \\ & u_{to} = 0 \text{ (displacement tangent to the edge)} \end{aligned}\right] \quad (44)$$

An alternative to $\frac{\partial w}{\partial n} = 0$ is the mathematically easier condition $\Psi_t = 0$ (zero rotation)

iii) Free Edge

$$\left.\begin{aligned} & Q_n = 0 \longrightarrow \psi_n = -\frac{\partial w}{\partial n} \\ & M_n = 0 \longrightarrow \frac{\partial \psi_t}{\partial t} = \frac{-1}{\nu_{tn}} \frac{\partial \psi_n}{\partial n} \\ & M_{nt} = 0 \longrightarrow \frac{\partial \psi_t}{\partial n} = -\frac{\partial \psi_n}{\partial t} \end{aligned}\right] \quad (45)$$

$N_n = 0 \longrightarrow \qquad \frac{\partial u_{n^o}}{\partial n} = -\nu_{tn} + \frac{\partial u_{t^o}}{\partial t}$

$N_n = 0 \longrightarrow \qquad \frac{\partial u_{no}}{\partial t} = -\frac{\partial u_{to}}{\partial n}$

Thick Plates on Elastic Foundations

Many models are used to represent the response of the elastic foundation to the overlying structures [8]. In this study, the soil resistance is modeled as follows:

1. For compressional resistance, linear Winkler model is used:

$$p(x,y) = K_z\, w(x,y) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (46)$$

   where $p(x,y)$ is the transverse reaction of the soil (per unit area) and K_z is termed the compressional foundation reaction.

2. For frictional resistance, the friction force (per unit area) can be represented either by a linear Winkler model or Coulomb model. When Winkler model is used, the frictional forces will be:

$$F_x(x,y) = -K_x u(x,y)_{(z=-h/2)} \quad F_y(x,y) = -K_y v(x,y)_{(z=-h/2)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(47)$$

where F_x or F_Y is the friction force per unit area in x or y direction, K_x or K_y is the frictional foundation reaction in x or y direction with units of stress per unit displacement and u $_{(Z=-h/2)}$ or v $_{(Z=-h/2)}$ is the horizontal displacement in x or y direction (at the bottom face).

The bottom face frictional forces F_x or F_y will develop moments μ_x or μ_z in xz or yz-plane:

$$\mu_x = \frac{h}{2}F_x \qquad \text{and} \qquad \mu_y = \frac{h}{2}F_y$$

Thus

$$\mu_x = \frac{h}{2}K_x u(x,y)_{(z=-h/2)} \quad \text{and} \quad \mu_y = \frac{h}{2}K_y v(x,y)_{(z=-h/2)} \qquad \ldots\ldots\ldots \quad (48)$$

But

$$u_{(z=-h/2)} = u_o - \psi_x \frac{h}{2} \qquad \text{and} \qquad \nu_{(z=-h/2)} = \nu_o - \psi_y \frac{h}{2}$$

Then

$$F_x = -K_x(u_0 - \psi_x \frac{h}{2}) \quad \text{and} \quad F_y = -K_y(\nu_0 - \psi_y \frac{h}{2}) \quad \text{............} \quad (49)$$

and

$$\mu_x = -K_x(u_0 - \psi_x \frac{h}{2}).\frac{h}{2} \quad \text{and} \quad \mu_y = -K_y(\nu_0 - \psi_y \frac{h}{2}).\frac{h}{2} \quad \text{..........}(50)$$

In the Coulomb friction model, the friction force (or sliding friction between two surfaces in contact) is independent of the value of horizontal displacement (or sliding) but is directly proportional to the normal reaction. Accordingly, the friction forces F_x or F_y could be related to the transverse deflection w as follows:

$$F_x = K_z\, w \tan(\delta_x) \qquad \text{or } F_y = K_z\, w \tan(\delta_y \qquad \text{.....................}(51)$$

where ($K_z\ w$) is the normal reaction of Winkler model, and $\delta_x$or δ_y is the angle of friction between the soil and the foundation in x or y-direction.

The direction of the friction force depends on the direction but not on the value of the horizontal displacement $u_{(z=-h/2)}$ or $\nu_{(Z=\_h/2)}$. So, it is mandatory to put a zero friction when there is no horizontal displacement. Accordingly equation (51) could be written as:

$$F_x = K_z\, w \tan(\delta_x)\ \{\varepsilon\} \quad \text{or} \qquad F_y = K_z\, w \tan(\delta_y)\ \{\varepsilon\} \quad \text{.........}(52)$$

where

$$\{\varepsilon\} = \begin{Bmatrix} -1 \ \ \text{when } u_{(z=-h/2)} \text{ or } \nu_{(Z=-h/2)} \text{ is negative} \\ 0 \ \ \text{when } u_{(z=-h/2)} \text{ or } \nu_{(Z=-h/2)} \text{ is zero} \\ +1 \ \ \text{when } u_{(z=-h/2)} \text{ or } \nu_{(z=-h/2)} \text{ is positive} \end{Bmatrix}$$

It should noted that u $_{(z=-h/2)}$ or v $_{(z=-h/2)}$ is positive at the bottom face when along the positive x or y-direction. According to Eq. (48), the distributed moments are:

$$\mu_x = \left(\frac{h}{2}\right) K_z w \tan(\delta_x)\{\varepsilon\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (53)$$

$$\mu_y = \left(\frac{h}{2}\right) K_z w \tan(\delta_y)\{\varepsilon\}$$

The Governing Equations for Thick Orthotropic Plates on Elastic Foundations

To solve the problems of plates on elastic foundations, the usual approach is based on the inclusion of the foundation reactions into the corresponding differential equations of plates.

The governing equations of thick orthotropic plates on elastic foundations characterized by Winkler model for both compressional and frictional resistances could be obtained by substituting equations (46), (49) and (50) into the equations (37) to (41). Thus, the governing equations will be:

$$c^2 G_{xz} h\left(\frac{\partial \psi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) + c^2 G_{yz} h\left(\frac{\partial \psi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2}\right) + q - k_z w = 0 \quad \ldots\ldots\ldots\ldots\ldots \quad (54)$$

$$D_x\left(\frac{\partial^2 \psi_x}{\partial x^2} + \nu_{yx}\frac{\partial^2 \psi_y}{\partial x \partial y}\right) + D_{xy}\left(\frac{\partial^2 \psi_x}{\partial y^2} + \frac{\partial^2 \psi_y}{\partial x \partial y}\right) - c^2 G_{xz} h\left(\psi_x + \frac{\partial w}{\partial x}\right) + K_x(u_0 - \psi_x \frac{h}{2})\frac{h}{2} = 0 \quad (55)$$

$$D_y\left(\frac{\partial^2 \psi_y}{\partial y^2} + \nu_{xy}\frac{\partial^2 \psi_x}{\partial x \partial y}\right) + D_{xy}\left(\frac{\partial^2 \psi_x}{\partial y \partial x} + \frac{\partial^2 \psi_y}{\partial x^2}\right) - c^2 G_{yz} h\left(\psi_y + \frac{\partial w}{\partial y}\right) + K_y(\nu_0 - \psi_y \frac{h}{2})\frac{h}{2} = 0 \quad (56)$$

$$\frac{E_x h}{1 - \nu_{xy}\nu_{yx}}\left(\frac{\partial^2 u_o}{\partial x^2} + \nu_{yx}\frac{\partial^2 v_o}{\partial x \partial y}\right) + G_{xy} h\left(\frac{\partial^2 u_o}{\partial y^2} + \frac{\partial^2 v_o}{\partial x \partial y}\right) - K_x(u_0 - \psi_x \frac{h}{2}) + \tau_{zx(h/2)} = 0 \quad (57)$$

$$\frac{E_x h}{1 - \nu_{xy}\nu_{yx}}\left(\frac{\partial^2 v_o}{\partial y^2} + \nu_{xy}\frac{\partial^2 u_o}{\partial x \partial y}\right) + G_{xy} h\left(\frac{\partial^2 u_o}{\partial x \partial y} + \frac{\partial^2 v_o}{\partial x^2}\right) - K_y(\nu_0 - \psi_y \frac{h}{2})\tau_{zy(h/2)} = 0 \quad (58)$$

New governing equations for thick orthotropic plates on elastic foundations characterized by Coulomb model for frictional resistance could be obtained by substituting equations (46), (52) and (53) into equations (37) to (41). Thus, the governing equations will be:

$$c^2G_{xz}h\left(\frac{\partial\psi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+c^2G_{yz}h\left(\frac{\partial\psi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)+q-k_z w=0 \qquad \ldots\ldots(59)$$

$$D_x\left(\frac{\partial^2\psi_x}{\partial x^2}+v_{yx}\frac{\partial^2\psi_y}{\partial x\partial y}\right)+D_{xy}\left(\frac{\partial^2\psi_x}{\partial y^2}+\frac{\partial^2\psi_y}{\partial x\partial y}\right)-c^2G_{xz}h\left(\psi_x+\frac{\partial w}{\partial x}\right)+K_z w\left(\frac{h}{2}\right)\tan(\delta_x)\{\varepsilon\}=0$$
(60)

$$D_y\left(\frac{\partial^2\psi_y}{\partial y^2}+v_{xy}\frac{\partial^2\psi_x}{\partial x\partial y}\right)+D_{xy}\left(\frac{\partial^2\psi_x}{\partial y\partial x}+\frac{\partial^2\psi_y}{\partial x^2}\right)-c^2G_{yz}h\left(\psi_y+\frac{\partial w}{\partial y}\right)+K_z w\left(\frac{h}{2}\right)\tan(\delta_y)\{\varepsilon\}=0$$
(61)

$$\frac{E_x h}{1-\nu_{xy}\nu_{yx}}\left(\frac{\partial^2 u_o}{\partial x^2}+\nu_{yx}\frac{\partial^2 v_o}{\partial x\partial y}\right)+G_{xy}h\left(\frac{\partial^2 u_o}{\partial y^2}+\frac{\partial^2 v_o}{\partial x\partial y}\right)+K_z w\tan(\delta_x)\{\varepsilon\}+\tau_{zx(h/2)}=0 \qquad (62)$$

$$\frac{E_y h}{1-\nu_{xy}\nu_{yx}}\left(\frac{\partial^2 v_o}{\partial y^2}+\nu_{xy}\frac{\partial^2 u_o}{\partial x\partial y}\right)+G_{xy}h\left(\frac{\partial^2 u_o}{\partial x\partial y}+\frac{\partial^2 v_o}{\partial x^2}\right)+K_z w\tan(\delta_y)\{\varepsilon\}+\tau_{zy(h/2)}=0 \qquad (63)$$

The main concept and mathematical formulation of the behavior of thick orthotropic plates under generalized loads resting on elastic foundations are presented. The equations are complex and intractable for direct solution. Therefore, numerical techniques are used.

Finite - Difference Method

In applying this method, the derivatives in the differential equations are replaced by differences at selected points. These points are located at the modes of a square or rectangular network (called finite-difference mesh). Therefore, all the governing differential equations are replaced by the equivalent difference equations. After this, the assembly for these equations is solved for the five degrees of freedom at each node. The stress resultants are obtained by back substitution of the resulting degrees of freedom into the equations of stress resultants after writing these equations in difference form. Outside fictitious nodes are needed to represent properly the boundary conditions, [8].

FINITE - ELEMENT METHOD

The finite-element method is also a numerical method for analysis of continuum structures. The basic philosophy of the finite element method is that the continuum is divided into small elements of various shapes, sizes and types which are then assembled together to form and approximate mathematical structure.

Certain functions are assumed to approximate the variation of the actual displacements over each finite element. The external loading is transformed into equivalent concentrated loadings at the nodes.

Herein, isoparametric 4-node rectangular elements with five independent degrees of freedom at each node are used[8] In this type, the same shape interpolation functions are used to describe the variation of displacements within the element and to specify the relation between the global (x,y) and the local (ξ,η) coordinate system. Also, each type of displacement at any point in the element is related to all displacements of same type at all nodes (no coupling or interaction as they are independent degrees of freedom). The external loads and moments and the foundation reactions are replaced by equivalent nodal forces by using the consistent method.

Applications

A square plate of side length (5m) and thickness (h=2m) is simply supported on the edges. The assumed elastic moduli are (E_x=25 kN/mm^2 E_y=5 kN/mm^2 E_z=15 kN/mm^2) and the assumed Poisson's ratios are (ν_{xy} = 0.75, ν_{xz} = 0.5 ν_{yz}= 0. 2). The plate is on an elastic foundation represented by Winkler model for compressional restraint with modulus *(K_z=10000 kN/m^3)* and for frictional restraint by either Winkler model with moduli *(K_x=K_y=20000 kN/m^3)* or Coulomb friction model with angles of friction (δ_X= δ_y =20 °). The loading was (q=25kN/m^2).

In order to use Coulomb friction model the sign of the horizontal displacements at the bottom face of the plate should be previously known at any point at that face. The sign of horizontal displacements cannot be estimated in case of complicated boundary conditions and complicated loadings but can be estimated in simple cases of symmetry in loading and boundary conditions. So, simple cases of loading and boundary conditions are considered.

For the simply supported plate with Winkler friction, Fig. (5) shows the deflection profiles along the center lines in the two perpendicular directions for thickness *(h=2m)* by the finite-element and the finite-difference methods. Fig. (6) shows the bending moment diagrams along the center lines in the two perpendicular directions. Fig. (7) shows the membrane force diagrams along the center lines in the two perpendicular directions. Fig.(8) shows the variation of central deflection with different thicknesses. The results show good agreements by these two methods especially for large thicknesses as shown in Fig.(8).

For the simply supported plate with the Coulomb model, same previous sequence of the figures are used for the results (Fig. (9) to Fig. (12)).

Effects of Elastic Foundations on the Plate Behavior

To show the effects of elastic foundations with both normal and frictional restraints (and consequently, distributed moments and shearing forces) in thick plates, the results of the central deflections in the simply supported plate with uniformly distributed loads are considered. The elastic foundation is represented by Winkler model for both compressional and frictional restraints.

In Figure (13), all the results of central deflections are related to that from the classical theory of plates in order to make comparison more general. Writing:

Cl = the central deflection from the classical theory of thin plates with no springs.

C2 = the central deflection from the classical theory of thin plates with transverse springs only.

C3 = the central deflection from thick plate theory with no springs.

C4 = the central deflection from thick plate theory with transverse springs only.

C5 = the central deflection from thick plate theory with transverse and horizontal springs.

Also writing:

$$g_1 = \frac{C2 - C1}{C2}$$

$$g_2 = \frac{C3 - C1}{C3}$$

$$g_3 = \frac{C4 - C1}{C4}$$

$$g_4 = \frac{C5 - C1}{C5}$$

The results are plotted according to the above parameters. In Figure (13), the variations of g_1, g_2, g_3 and g_4 with thickness are plotted. The following points may be concluded from this figure:

1. Graph (g_1) shows the effects of the elastic foundation on the classical thin plate theory. This effect is considerable when the plate is very thin. The effect diminishes when the plate becomes stiffer (when deflections are small).

2. Graph (g_2) represents the difference between the classical thin plate solution and
Mindlin's thick plate solution. The latter can be obtained by elimination of the spring terms from the governing equations. The thick plate theory gives higher deflections than the thin plate theory due to the contribution of transverse shear deformation. The difference becomes more considerable for higher thicknesses.
3. Graphs (g_3) and (g_4) show the effects of elastic foundations on the thick plate behavior. Graph (g_3) can be produced by eliminating friction terms from the governing equations. The graphs (g_3) and (g_4) are almost coinciding. This indicates that the effect of friction at soil-plate interface on thick plate deflection is small and can be neglected.

The graphs g_2, g_3 and g_4 coincide when the plate becomes very stiff. This indicates that the effect of the elastic foundation is small when the plate is very stiff.

Since the compressional and the frictional restraints are related to the transverse and longitudinal displacements, therefore the effects of these restraints will diminish for very stiff plates (small transverse and longitudinal displacements), although the friction induced moments are proportional to the thickness (h) (Eqs. (50)).

To study the effect of variation of thickness on the membrane forces in plates, a simply supported plate with uniformly distributed load and resting on Winkler compressional and frictional foundation is considered. The results are presented in Fig. (14) which gives the membrane force diagram along the center line in x-direction. Fig. (14) shows that the membrane forces decrease with increasing of the thickness. The membrane forces are proportional to the bottom face shearing forces which in turn proportional to the horizontal displacements at the bottom face of the plate (Winkler model Eq. (47)) and to the vertical deflection (Coulomb model Eq.(51)). When the thickness increases, the stiffness will increase causing a decrease in the horizontal and vertical displacements and accordingly a decrease in the membrane forces. Although the friction forces at plate-soil interface are proportional to thickness (h) (Eq.(49)), the horizontal and vertical displacements are the dominant parameters.

Effect of Type and Magnitude of Loading and Boundary Conditions on the Plate Behavior

To show the effect of type and magnitude of loading and boundary conditions on the contribution of shearing deformation (the percentage of the difference in central deflections between thick and thin plate solutions), plates with simply supported, fixed and free edges are considered. The simply-supported and fixed-edge plates are loaded by a uniformly distributed load *(q=25 kN/m^2)* or by a concentrated central load (P= l00 kN). The free-edge plate is loaded by a concentrated central load (P=100 kN). The results are shown in Fig.(15). From this figure, the following remarks can be deduced:

1. The percentage of error introduced by neglecting the transverse shearing
   deformations increases with increasing of depth (thickness to span ratio).
2. The fixed edge plates are shear deformable more than the simply supported plates. The effect of shearing deformations on free-edge plates is less than the other two types. This indicates that more restraints on the plate make the plate more affected by transverse shearing deformations.

3. In cases of concentrated loads, the influence of transverse shearing deformation is found to be greater than the influence in cases of uniformly distributed loads. Concentrated loads give high transverse shearing forces over large portions of the plate.
4. When the values of the uniformly distributed and concentrated loads are varied, the same five curves are obtained exactly. This indicates that the percentage of difference in central deflections between thick and thin plate solutions is independent on the value of loading for same properties of plates and elastic foundations in each case of loading. Also, the same five curves are obtained exactly when the span and thickness are varied but for constant thickness to span ratio.

CONCLUSIONS

1. Good agreements are obtained between the finite-difference and finite-element methods. The results show that the two methods are almost identical especially for large thicknesses. Obvious differences are noticed in small thicknesses (probably due to shear and membrane locking).
2. The effect of shearing forces at the plate-foundation interface and accordingly, the effect of distributed moments are small on transverse deflections of plates and on stress resultants.
3. The influence of transverse shearing deformation is greater for concentrated loads than for distributed loads and greater in fixed-edge plates than in simply supported plates. The free-edge plates are influenced by transverse shearing deformations by a magnitude less than in fixed-edge or simply supported plates.

REFERENCES

1. Timoshenko,S. and Winowsky-Krieger, S., "Theory of Plates and Shellls", McGraw-Hill 1959.
2. Reissner, E., 'The effect of transverse shear deformation on the bending of elastic plates", ASME, Journal of Applied Mechanics, Vol. 12, 1945, PP. 69-77.
3. Mindlin, R., "Influence of rotatory inertia and shear on flexural motions of isotropic elastic plates", ASME, Journal of Applied Mechanics, Vol.18 March 1951, PP. 31-38.
4. Schmidt, R., "A refined nonlinear theory of plates with transverse shear deformation", Journal of Ind. Math. sec. 27Pt, Vol 1, 1977 P.23.
5. Levinson, M., "An accurate, simple theory of statics and dynamics of elastic plates", Mechanics Research Communications, Vol. 7, 1980, PP. 343-350.
6. Husain, H.M. "Thick isotropic plates under generalized loads" Journal of Engineering and Development, College of Engineering, Mustansiriyah University, Baghdad, vol.1 No.1 1966
7. Jones, R. M., "Mechanics of Composite Materials", McGraw-Hill, U.S.A., 1975.
8. Al-Mahdi, Abdulameer Radhi, "Thick Orthotropic Rectangular Plates on Elastic Foundations", M.Sc. Thesis, Saddam University, Dec. 1994.

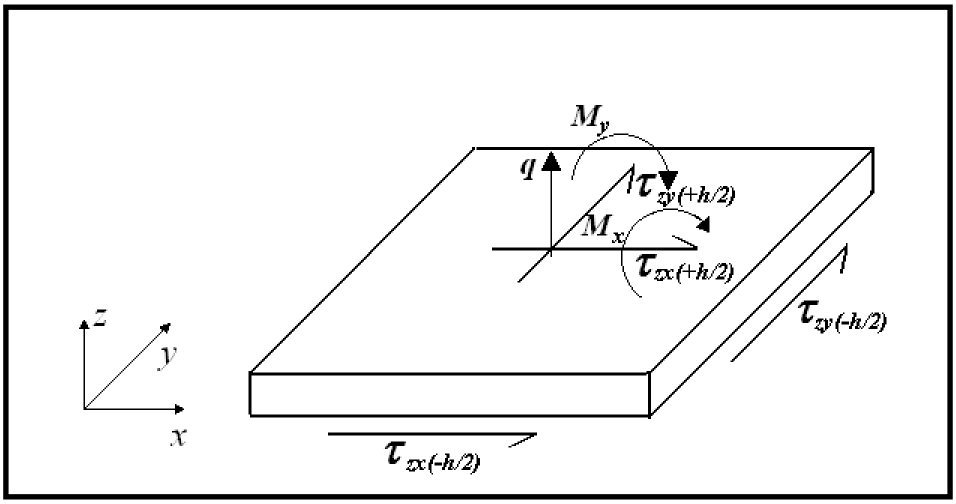

Fig. (1) Thick plate element under generalized loading

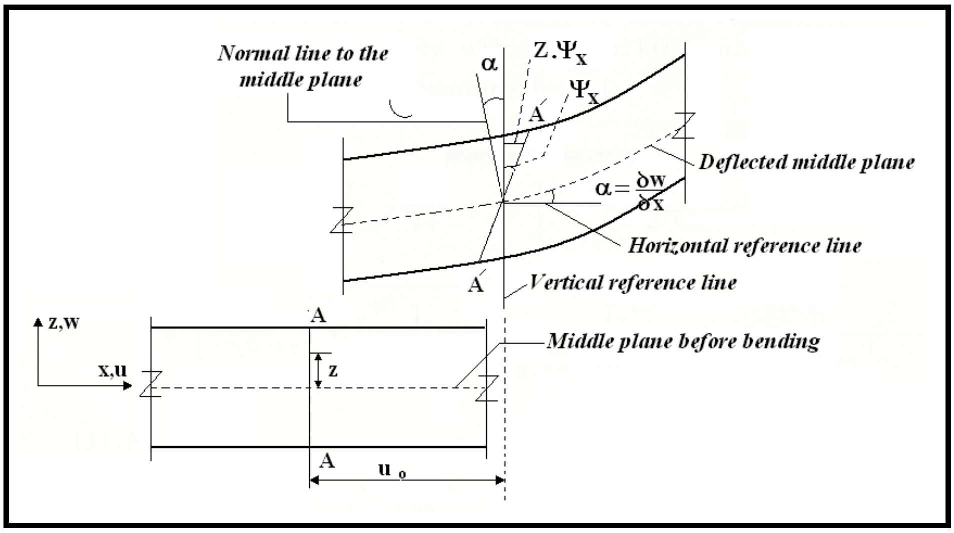

Fig. (2) Deformation of thick plate section

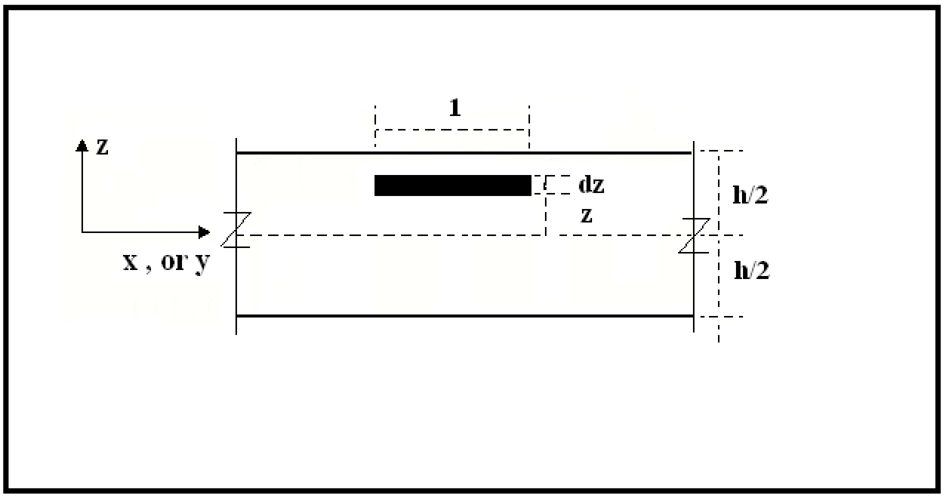

Fig. (3)

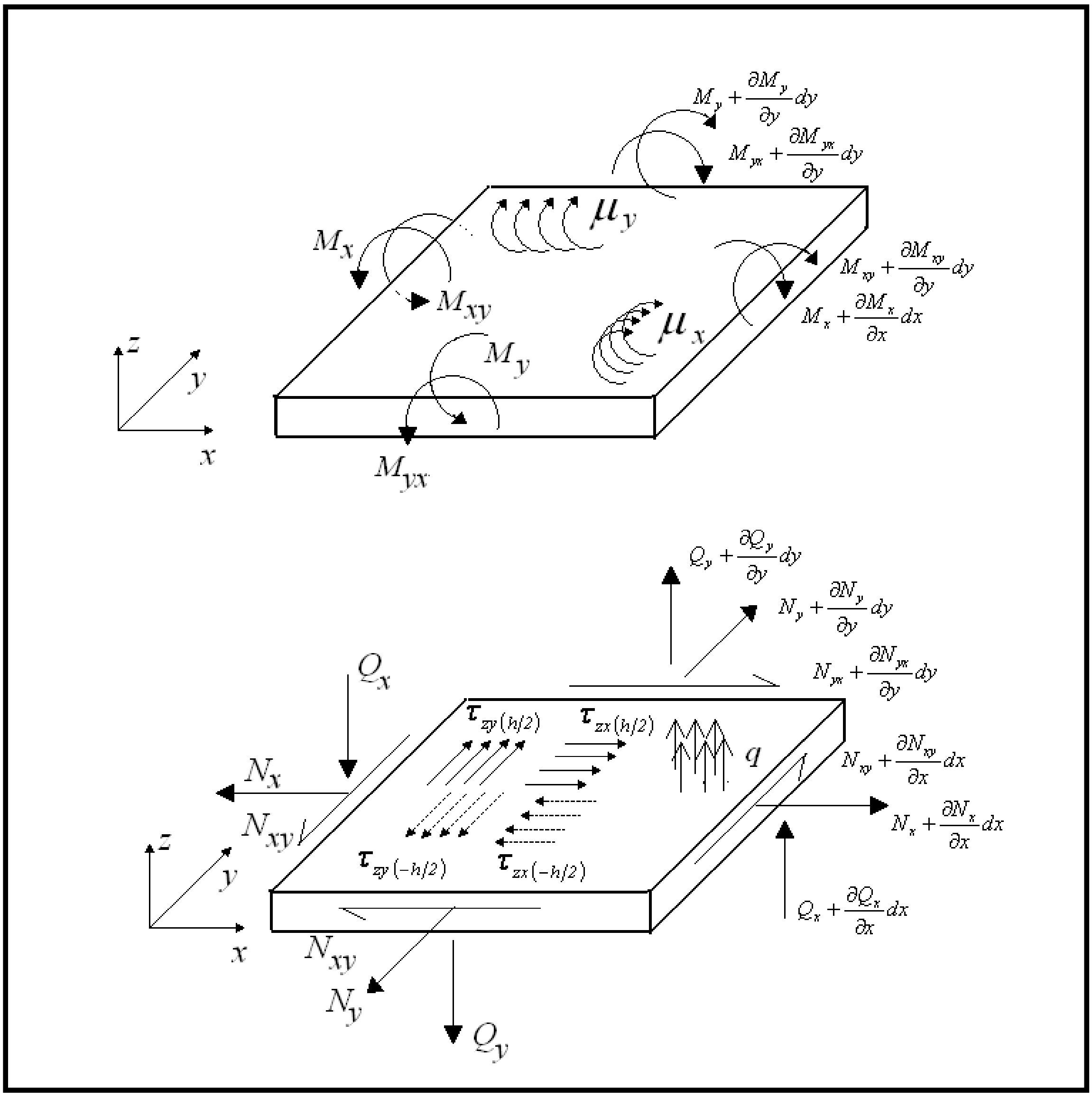

Fig. (4) Applied and resulting moments and forces

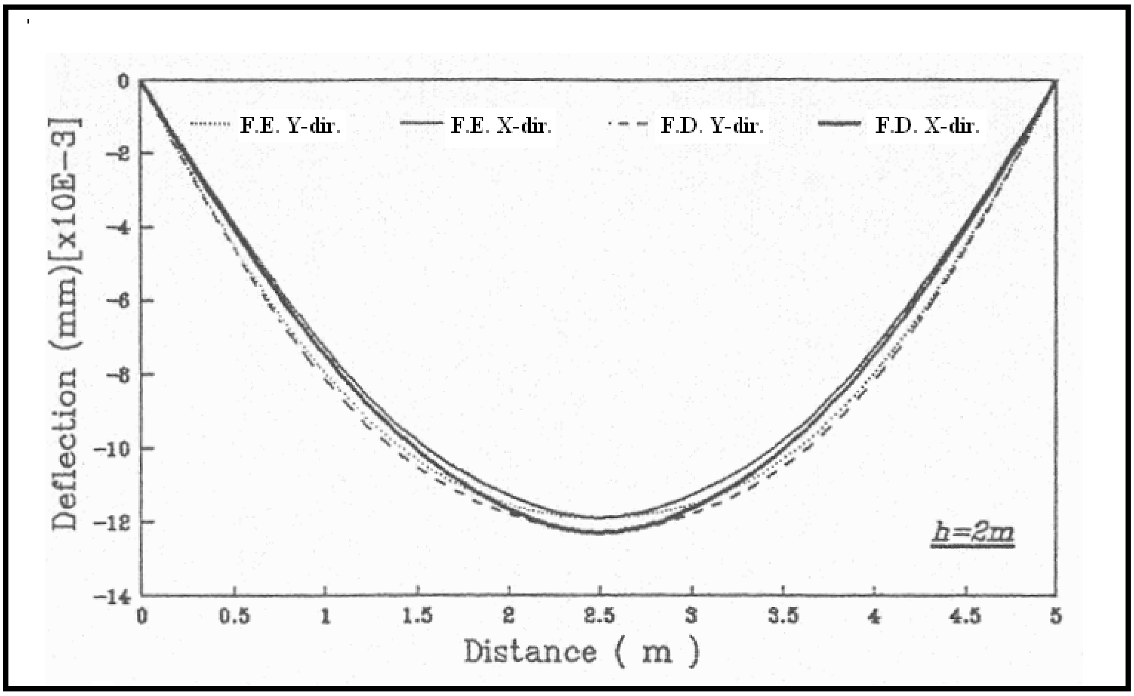

Fig. (5) Deflection profiles in two perpendicular directions for a simply supported thick plate [Winkler friction model]

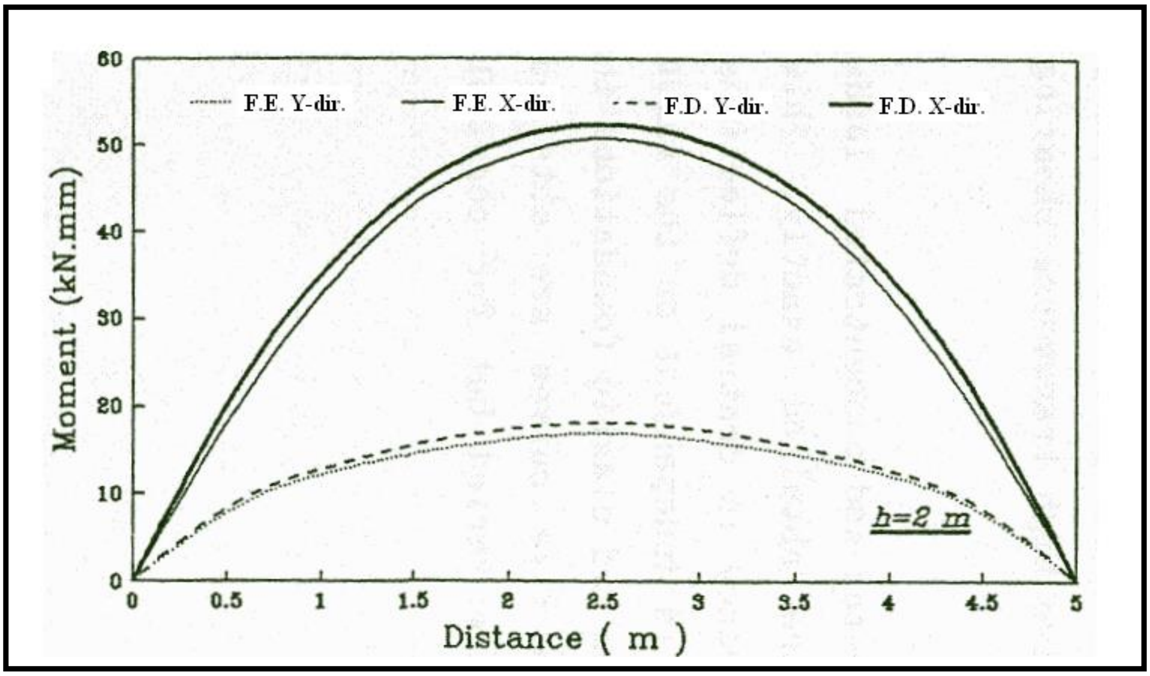

Fig. (6) B.M Diagrams in two perpendicular directions for a simply supported thick plate [Winkler friction model]

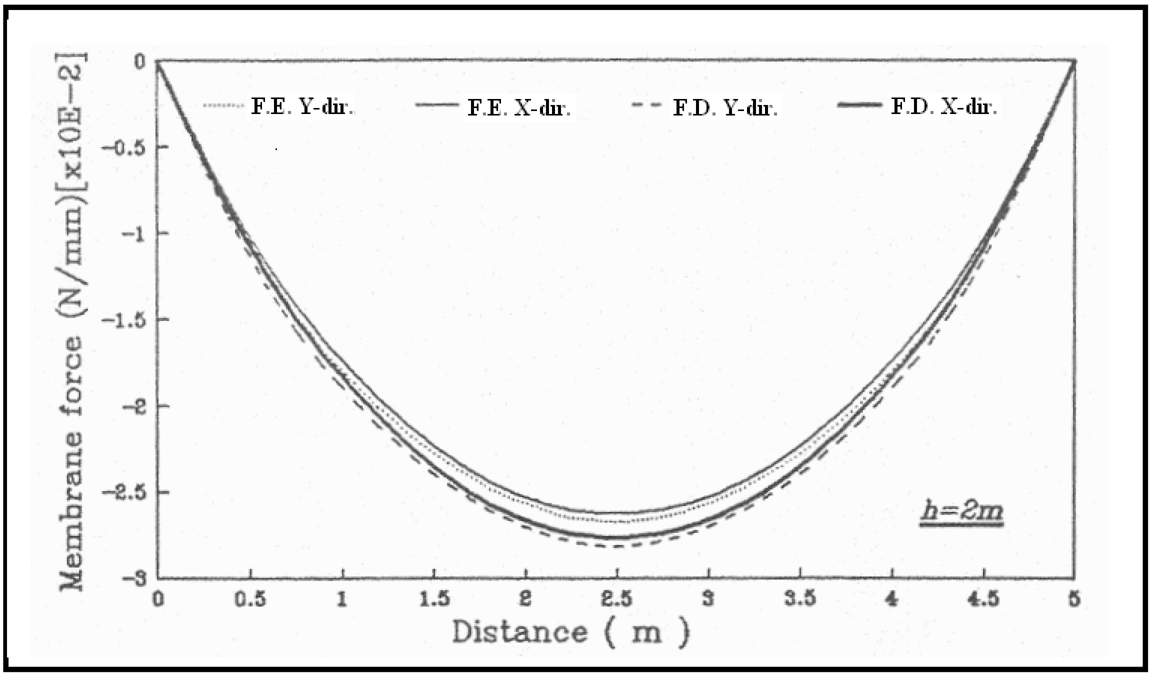

Fig. (7) Membrane force diagram in two perpendicular directions for a simply supported thick plate [Winkler friction model]

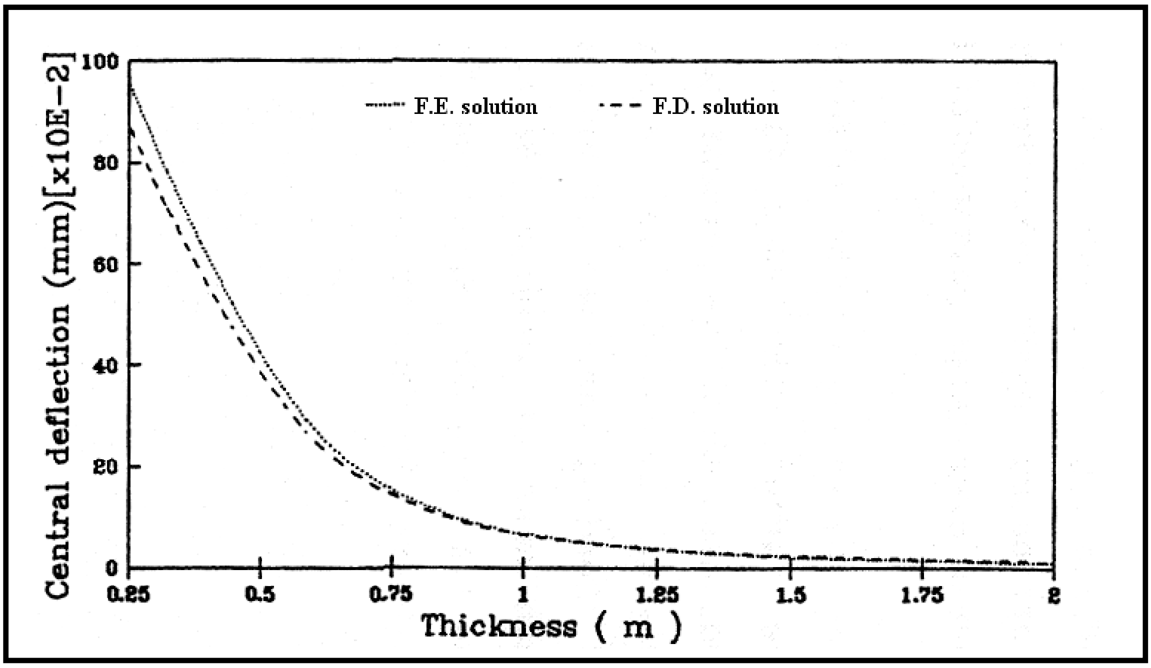

Fig. (8) Central deflection simply supported plate of various thickness [Winkler friction model]

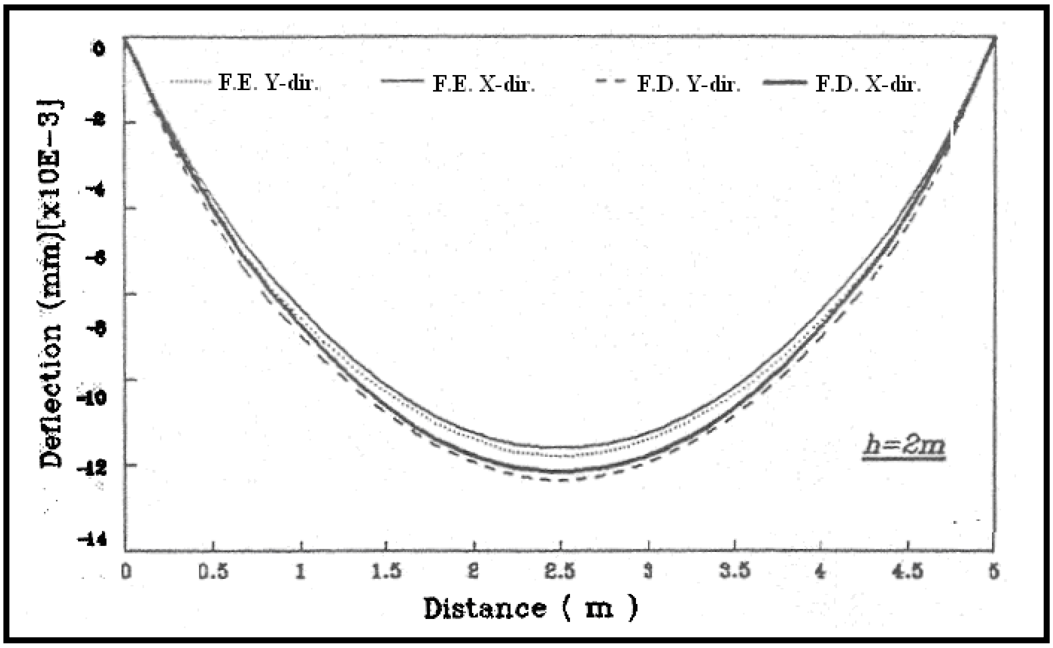

Fig. (9) Deflection profiles in two perpendicular directions for a simply supported thick plate [Coulomb friction model]

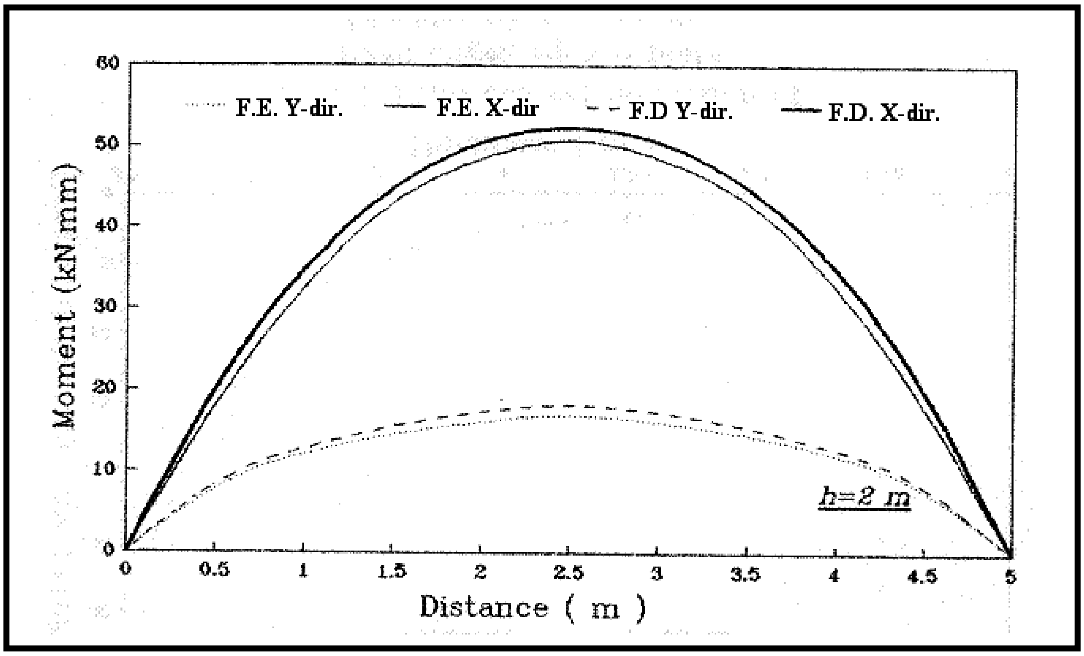

Fig. (10) B.M. diagrams in two perpendicular directions for a simply supported thick plate [Coulomb friction model]

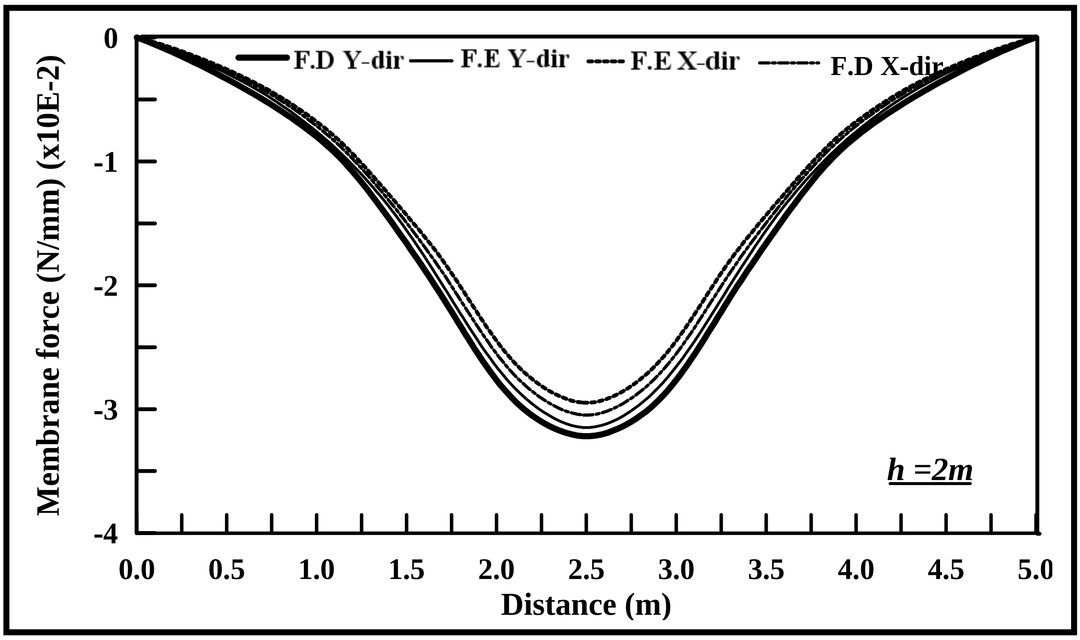

Fig. (11) Membrane force diagrams in two perpendicular directions for a simply supported thick plate [Coulomb friction model]

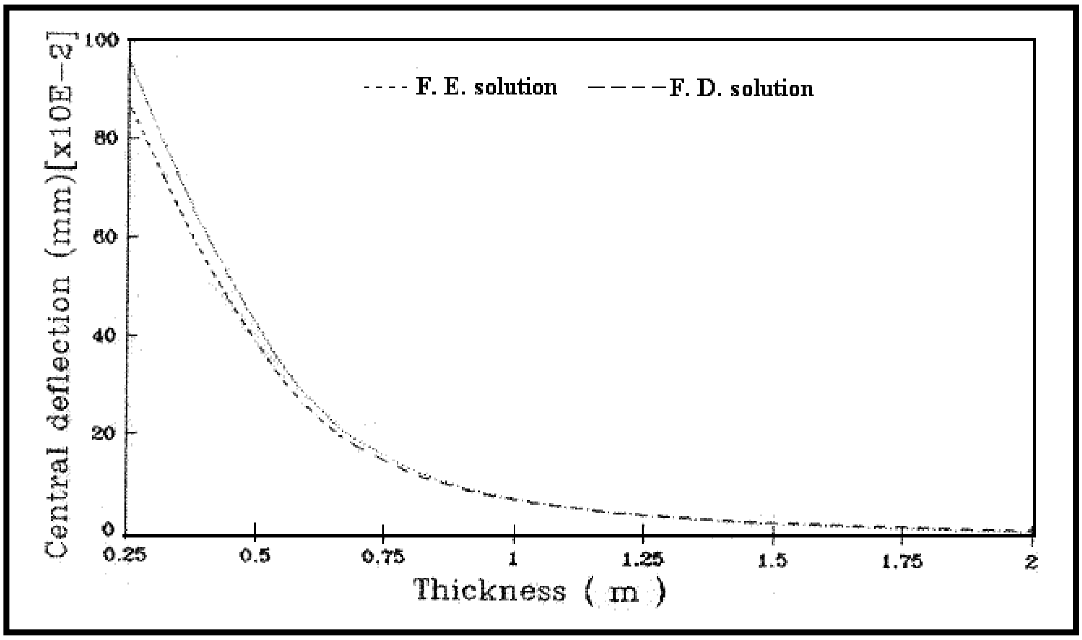

Fig. (12) Central deflection of a simply supported plate of various thicknesses [Coulomb friction model]

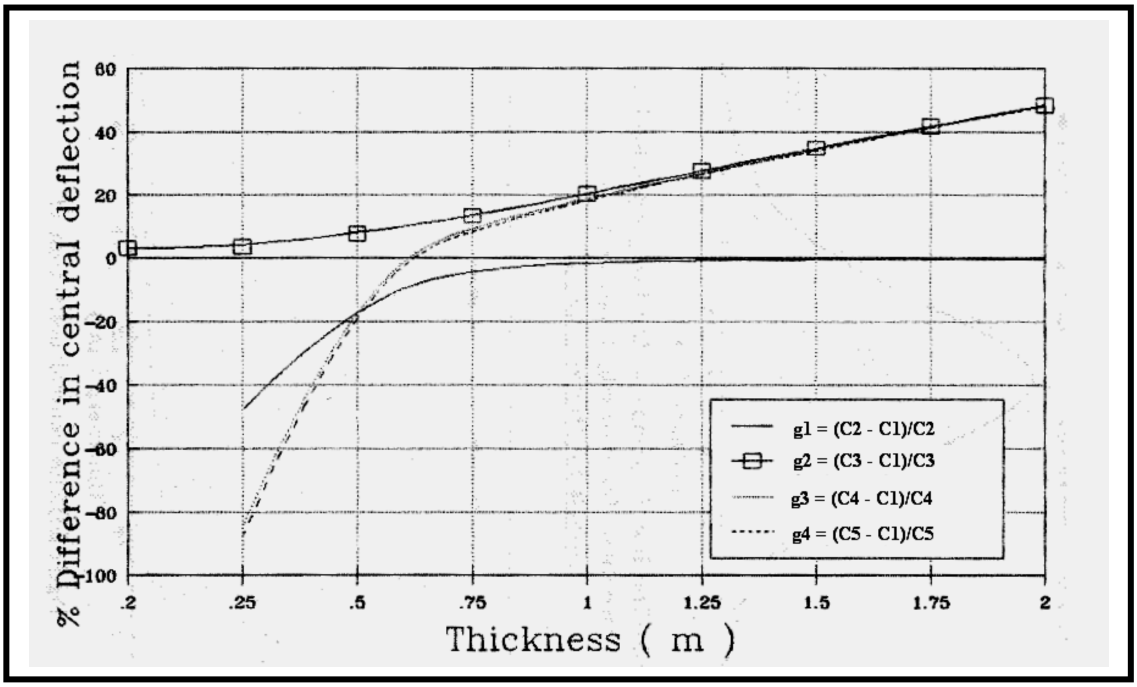

Fig. (13) Effect of compressional and frictional Winkler foundation on central deflection of thick plate

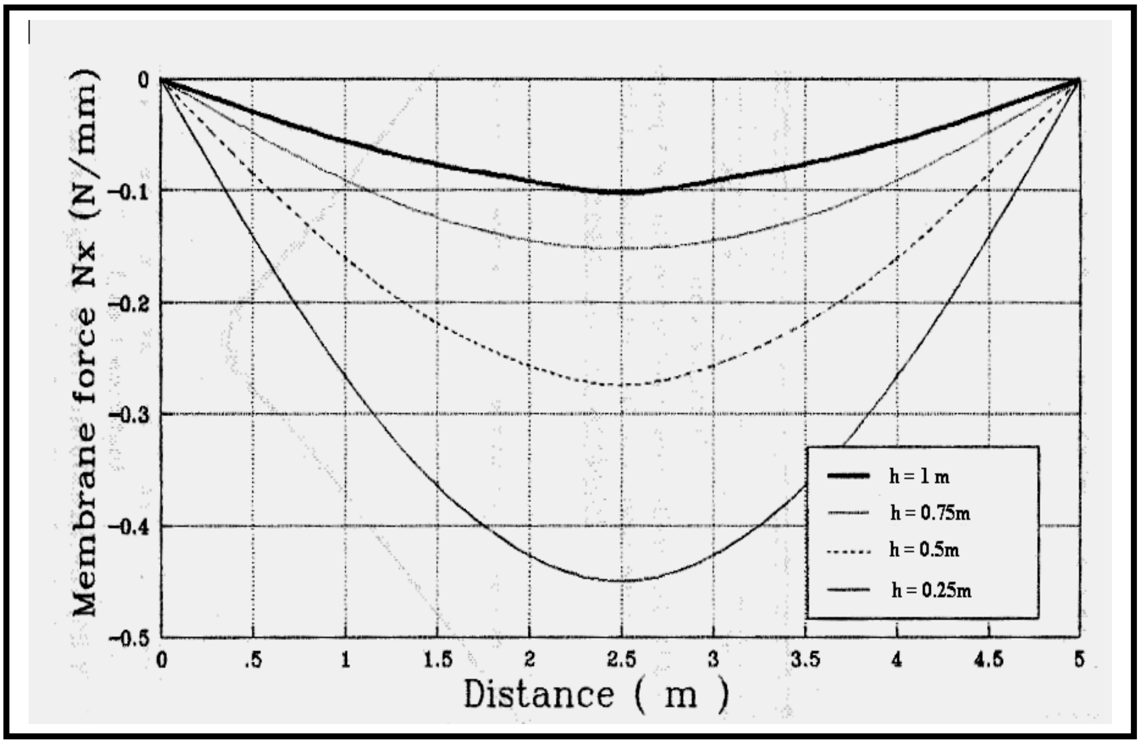

Fig. (14) Effect of thickness on membrane forces in a thick plate on elastic foundation with Winkler friction model

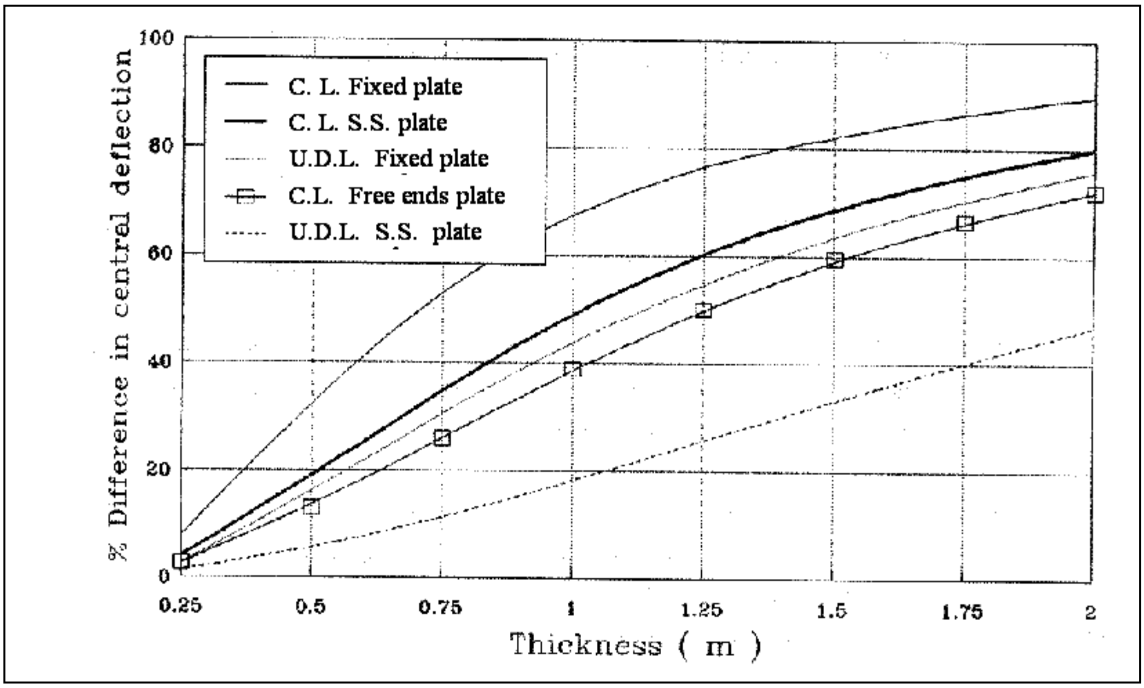

Fig. (15) Effect of type and magnitude of loading and boundary conditions on the percentage of the difference in the central deflection between thick and thin plate solutions

البلاطات السميكة المستطيلة المختلفة الخواص بالاتجاهين المستندة على اسس مرنة

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الخلاصة

في هذا البحث تم تطوير نظرية مندلين (Mindlin) للبلاطات السميكة لاستخدامها في البلاطات السميكة المختلفة الخواص بالاتجاهين (orthotropic) المعرضة لتأثير العزوم و قوى القص الموزعة خارجيا على الوجه الاعلى و الاسفل للبلاطة. قوى القص هذه تنتج قوى مستوية سطحية (غشائية) في البلاطات حيث تم اخذ التأثيرات التمددية لهذه القوى في التحليل. خمس درجات من الحرية اخذت عند تحليل المقاطع المستعرضة للبلاطة ، تضمنت درجة للهطول المستعرض و درجتان للدوران المستقلة من ا[illegible] بيعي الى منتصف البلاطة و كذلك درجتان (1-34) للازاحتان الغشائيتين العموديتين المتبادلتين. ـــس ـعادلات استخدمت لتمثيل البلاطة السميكة المختلفة الخواص متضمنة تأثير العزوم الخارجية و قوى القص المسلطة و الموزعة عليها مع الاخذ بنظر الاعتبار المقيدات الانظغاطية و الاحتكاك

و لغرض التطبيق فقد تم اختيار تحليل بلاطة سميكة مختلفة الخواص بالاتجاهين مستندة على اساس مرن (elastic foundation) و تحت تأثير عزوم و قوى قص.

استخدمت طريقة الفروق المحددة لحل المعادلات اضافة الى طريقة العناصر المحددة و حيث وجد بان النتائج متقاربة و بشكل جيد و لكلا الطريقتين

الكلمات الدالة

الاسس المرنة ، الفروق المحددة ، العناصر المحددة، البلاطات المختلفة الخواص بالاتجاهين، البلاطات السميكة